Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

1 - 5 Doolittle’s method
Show the factorization and solve by Doolittle’s method.

Okay, first there seems to be disagreement about decomposing. **First Version:** I am looking at the matrix in example 1 on p. 853, the matrix being m={{3,5,2},{0,8,2},{6,2,8}}. For the Doolittle decomposed matrix upper, the text comes up with m={{3,5,2},{0,8,2},{0,0,6}} as the upper diagonal matrix, and m={{1,0,0},{0,1,0},{2,-1,1}} as the lower diagonal matrix. Using Mathematica **LUDecomposition** results in m={{3,5,2},{0,8,2},{2,-1,6}}. This answer is coded into a composite upper and lower diagonal matrix. When properly decoded, the lower is the same as the text lower, and so the pairs are equal. Thus Doolittle Text is equal to Mathematica LUD, which is what I wanted to know. On-line discussion and timing tests show LUD is very fast. I think I will make it my default, anyway for the present. There are at least 3 Mathematica modules discussed at *https://mathematica.stackexchange.com/questions/78700/how-to-speed-up-auxilary-doolittledecomposite-function* and they all agree with **LUDecomposition**. **Second Version:** There is an on-line matrix decomposer at *https://www.easycalculation.com/matrix/lu-decomposition-matrix.php*, and it returns m={{6,2,8},{0,8,2},{0,0,-3}} for upper. This version of the decomposure is also the one returned at the site *https://keisan.casio.com/exec/system/15076953047019*.

So can I reconstruct the original by dotting the lower with upper? Second version above, first

```
m = {{3, 5, 2}, {0, 8, 2}, {6, 2, 8}};
u = {{6, 2, 8}, {0, 8, 2}, {0, 0, -3}};
l = {{1, 0, 0}, {0, 1, 0}, {.5, .5, 1}};
```

```
l.u
```

```
{{6., 2., 8.}, {0., 8., 2.}, {3., 5., 2.}}
```

This is sort of like the original, except with upper and lower rows switched. I don’t know if it is right or wrong, but both the last two sites got the same thing, and, one being Casio, I tend to give them some credit. Next, First version, **LUDecomposition** by Mathematica and the SEMma question

```
u = {{3, 5, 2}, {0, 8, 2}, {0, 0, 6}};
l = {{1, 0, 0}, {0, 1, 0}, {2, -1, 1}};
```

```
l.u
{{3, 5, 2}, {0, 8, 2}, {6, 2, 8}}
```

That looks good, restoring the original matrix exactly. With that out of the way, I can go on to the actual problems for this problem section.

1. $4 x_1 + 5 x_2 == 14$; $12 x_1 + 14 x_2 == 36$

```
m = {{4, 5}, {12, 14}}
n = {{14}, {36}}
{{4, 5}, {12, 14}}
{{14}, {36}}
```

```
LUDecomposition[m]
{{{4, 5}, {3, -1}}, {1, 2}, 0}
```

After decoding, this would be u={{4,5},{0,-1}}, l={{1,0},{3,1}}

```
u = {{4, 5}, {0, -1}};
l = {{1, 0}, {3, 1}};
```

```
l.u
{{4, 5}, {12, 14}}
```

Successfully reconstituted.

```
LinearSolve[m, n]
{{-4}, {6}}
```

Both factorization and multiplication agree with the text answer.

3. $5 x_1 + 4 x_2 + x_3 == 6.8$;
$10 x_1 + 9 x_2 + 4 x_3 == 17.6$; $10 x_1 + 13 x_2 + 15 x_3 == 38.4$

Since this subsection of problems was supposed to utilize Doolittle and not LUD, I will insert a Doolittle module and credit it to 2012rcampion as responder to the question on SEMma, *https://mathematica.stackexchange.com/questions/78700/how-to-speed-up-auxilary-doolittlede-composite-function.* It seems to have the same functionality as LUD, except that it does not include pivoting and condition information.

```
doolittleDecomposite2[mat_?MatrixQ] :=
 Module[{temp = ConstantArray[0, Dimensions@mat], row = Length@mat},
  Do[Do[temp[[k, j]] =
      mat[[k, j]] - temp[[k, ;; k - 1]].temp[[ ;; k - 1, j]], {j, k, row}];
   Do[temp[[i, k]] = (mat[[i, k]] - temp[[i, ;; k - 1]].
          temp[[ ;; k - 1, k]]) / temp[[k, k]], {i, k + 1, row}];, {k, row}];
  temp]
```

```
m = {{5, 4, 1}, {10, 9, 4}, {10, 13, 15}};
n = {{6.8}, {17.6}, {38.4}};
```

```
doolittleDecomposite2[m]
```

```
{{5, 4, 1}, {2, 1, 2}, {2, 5, 3}}
```

Decoding, I make it out as

```
u = {{5, 4, 1}, {0, 1, 2}, {0, 0, 3}};
l = {{1, 0, 0}, {2, 1, 0}, {2, 5, 1}}
```

```
l.u
```

```
{{5, 4, 1}, {10, 9, 4}, {10, 13, 15}}
```

Success in reconstituting.

```
LinearSolve[m, n]
```

```
{{0.4}, {0.8}, {1.6}}
```

Both factorization and multiplication agree with the text answer.

5. $3 x_1 + 9 x_2 + 6 x_3 == 4.6$;
$18 x_1 + 48 x_2 + 39 x_3 = 27.2$; $9 x_1 - 27 x_2 + 42 x_3 == 9.0$

```
m = {{3, 9, 6}, {18, 48, 39}, {9, -27, 42}};
n = {{4.6}, {27.2}, {9.0}};
```

```
doolittleDecomposite2[m]
```

```
{{3, 9, 6}, {6, -6, 3}, {3, 9, -3}}
```

Decoding, what I see is

```
u = {{3, 9, 6}, {0, -6, 3}, {0, 0, -3}};
l = {{1, 0, 0}, {6, 1, 0}, {3, 9, 1}};
```

```
l.u
```

```
{{3, 9, 6}, {18, 48, 39}, {9, -27, 42}}
```

Success in reconstituting.

```
LinearSolve[m, n]
```

```
{{-0.0666667}, {0.266667}, {0.4}}
```

```
N[{-1 / 15, 4 / 15}]
{-0.0666667, 0.266667}
```

Both factorization and multiplication agree with the text answer.

7 - 12 Cholesky’s method
Show the factorization and solve.

7. $9 x_1 + 6 x_2 + 12 x_3 == 17.4$;
$6 x_1 + 13 x_2 + 11 x_3 == 23.6$; $12 x_1 + 11 x_2 + 26 x_3 == 30.8$

Mathematica has a function for Cholesky decomposition, but it returns only the upper diagonal matrix. Let’s see what effect that may have.

```
m = {{9, 6, 12}, {6, 13, 11}, {12, 11, 26}};
n = {{17.4}, {23.6}, {30.8}};
```

```
cd = CholeskyDecomposition[m]
```

```
{{3, 2, 4}, {0, 3, 1}, {0, 0, 3}}
```

The returned expression from CD being the upper diag matrix, I need to use the **ConjugateTranspose** to check it

```
ct = ConjugateTranspose[%]
```

```
{{3, 0, 0}, {2, 3, 0}, {4, 1, 3}}
```

```
ct.cd
{{9, 6, 12}, {6, 13, 11}, {12, 11, 26}}
```

It looks good. And it’s interesting, I don’t think I ever ‘percentaged’ up through a text cell before.

```
LinearSolve[m, n]
```

```
{{0.6}, {1.2}, {0.4}}
```

9. $0.01 x_1 + 0 + 0.03 x_3 == 0.14$;
$0 + 0.16 x_2 + 0.08 x_3 == 0.16$; $0.03 x_1 + 0.08 x_2 + 0.14 x_3 == 0.54$

```
m = {{0.01, 0, 0.03}, {0, 0.16, 0.08}, {0.03, 0.08, 0.14}};
n = {{0.14}, {0.16}, {0.54}};
```

```
cd = CholeskyDecomposition[m]
```

```
{{0.1, 0., 0.3}, {0., 0.4, 0.2}, {0., 0., 0.1}}
```

The returned expression from CD being the upper diag mat, I need to use the **ConjugateTranspose** to check it

```
ct = ConjugateTranspose[%]
```

```
{{0.1, 0., 0.}, {0., 0.4, 0.}, {0.3, 0.2, 0.1}}
```

```
ct.cd
{{0.01, 0., 0.03}, {0., 0.16, 0.08}, {0.03, 0.08, 0.14}}
```

The reconstitution was successful.

```
LinearSolve[m, n]
```

```
{{2.}, {-1.}, {4.}}
```

Green above agrees with text. As for the multiplication, the text answer is {{2.},{-11.},{4.}}. I don't see how that can be reconciled with the other matrix attributes, so I assume it is a typo. (I also got verification from WolframAlpha for the yellow answer.)

11. $x_1 - x_2 + 3x_3 + 2x_4 == 15$; $-x_1 + 5x_2 - 5x_3 - 2x_4 == -35$;
$3x_1 - 5x_2 + 19x_3 + 3x_4 == 94$; $2x_1 - 2x_2 + 3x_3 + 21x_4 == 1$

```
m = {{1, -1, 3, 2}, {-1, 5, -5, -2}, {3, -5, 19, 3}, {2, -2, 3, 21}};
n = {{15}, {-35}, {94}, {1}};
```

```
cd = CholeskyDecomposition[m]
```

```
{{1, -1, 3, 2}, {0, 2, -1, 0}, {0, 0, 3, -1}, {0, 0, 0, 4}}
```

Here I need to use the **ConjugateTranspose** to check it

```
ct = ConjugateTranspose[%]
```

```
{{1, 0, 0, 0}, {-1, 2, 0, 0}, {3, -1, 3, 0}, {2, 0, -1, 4}}
```

```
ct.cd
{{1, -1, 3, 2}, {-1, 5, -5, -2}, {3, -5, 19, 3}, {2, -2, 3, 21}}
```

The reconstitution was successful.

```
LinearSolve[m, n]
```

```
{{2}, {-3}, {4}, {-1}}
```

The last problem demonstrated that a 4x4 matrix problem is no more time-consuming than

3x3.

13. Definiteness. Let A, B be $n \times n$ and positive definite. Are $-A$, A^T, $A + B$, $A - B$ positive definite?

I would say no, yes, yes, no. The rationale comes from looking at the pivots. The site *https://www.gaussianwaves.com/2013/04/tests-for-positive-definiteness-of-a-matrix/* gives some easy-to-use info to test matrices for positive-definiteness. I am assuming the information is correct. Note that first of all, the matrix in question has to be symmetric. Why did I give the answers I gave? Hey, I just checked, and all answers are right! Okay, nr 1., multiplying by -1 immediately negates all positive pivots, ruling out the possibility of positive-definiteness. Nr 2., With transpose-ization, entries in the matrix may be multiplied by fractions, but the sign will stay the same, thus if A is positive-definite, then so should A^T be also. Third and fourth are yes and no because A+B preserves all the positive signs in both, whereas A-B could ruin positive definiteness for A if any pivot entries in B are larger than the site they will be subtracted from in A.

15 - 19 Inverse
Find the inverse by the Gauss-Jordan method, showing the details.

15. In problem 1.

From problem 1

```
4 x1 + 5 x2 == 14 ;  12 x1 + 14 x2 == 36
12 x1 + 14 x2 == 36
```

```
m = {{4, 5}, {12, 14}}
```

```
{{4, 5}, {12, 14}}
```

```
Inverse[m]
```

$$\left\{\left\{-\frac{7}{2}, \frac{5}{4}\right\}, \{3, -1\}\right\}$$

The green cell above matches the answer in the text. Obviously not done by Gauss-Jordan, but the inverse just the same.

17. In Team Project 6 (c).

From 6(c), use Doolittle’s to factorize

```
m = {{1, -4, 2}, {-4, 25, 4}, {2, 4, 24}};
n = {{17.4}, {23.6}, {30.8}};
```

```
doolittleDecomposite2[mat_?MatrixQ] :=
 Module[{temp = ConstantArray[0, Dimensions@mat], row = Length@mat},
  Do[Do[temp[[k, j]] =
      mat[[k, j]] - temp[[k, ;; k - 1]].temp[[ ;; k - 1, j]], {j, k, row}];
   Do[temp[[i, k]] = (mat[[i, k]] - temp[[i, ;; k - 1]].
         temp[[ ;; k - 1, k]]) / temp[[k, k]], {i, k + 1, row}];, {k, row}];
  temp]
```

```
doolittleDecomposite2[m]
```

$\{\{1, -4, 2\}, \{-4, 9, 12\}, \{2, \frac{4}{3}, 4\}\}$

```
LUDecomposition[m]
```

$\{\{\{1, -4, 2\}, \{-4, 9, 12\}, \{2, \frac{4}{3}, 4\}\}, \{1, 2, 3\}, 0\}$

```
Inverse[m]
```

$\{\{\frac{146}{9}, \frac{26}{9}, -\frac{11}{6}\}, \{\frac{26}{9}, \frac{5}{9}, -\frac{1}{3}\}, \{-\frac{11}{6}, -\frac{1}{3}, \frac{1}{4}\}\}$

PossibleZeroQ$[\frac{146}{9} - \frac{584}{36}]$

True

The expression in green is equivalent to the answer in the text, after an alternate denominator pattern is identified by the PZQ.

19. In problem 12.

```
m = {{4, 2, 4, 0}, {2, 2, 3, 2}, {4, 3, 6, 3}, {0, 2, 3, 9}};
n = {{20}, {36}, {60}, {122}};
```

```
Inverse[m]
```

$\{\{\frac{21}{16}, -\frac{3}{8}, -\frac{7}{8}, \frac{3}{8}\}, \{-\frac{3}{8}, \frac{9}{4}, -\frac{3}{4}, -\frac{1}{4}\},$
$\{-\frac{7}{8}, -\frac{3}{4}, \frac{5}{4}, -\frac{1}{4}\}, \{\frac{3}{8}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\}\}$

The expression in the green cell above matches the answer in the text.